Math 32A, Lecture 1
Multivariable Calculus

Midterm 2

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books, notes, or calculators. Show all your work; partial credit will be given for progress toward correct solutions, but unsupported correct answers will not receive credit. Remember to make your drawings large and clear, and to label your axes.

Write your solutions in the space below the questions. If you need more space, use the back of the page. Do not turn in your scratch paper.

Name: Solutions

UID: ____________________

Section: ____________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Problem 1.

Consider the space curve $\mathbf{r}(s) = \langle \frac{1}{3}(1+s)^{\frac{3}{2}}, \frac{1}{3}(1-s)^{\frac{3}{2}}, \frac{s}{\sqrt{2}} \rangle$.

(a) [3pts.] Show that $\mathbf{r}(s)$ is a unit speed curve.

(b) [4pts.] Find the curvature of $\mathbf{r}(s)$ at $s = 0$, and the unit normal vector $\mathbf{N}$ to the curve at this point. [Hint: In light of part (a), there is a fast way to do this.]

(c) [3pts.] Find the osculating plane to $\mathbf{r}(s)$ at $s = 0$, and say what it represents geometrically.

(a) $\vec{r}\,'(s) = \langle \frac{1}{2}(1+s)^{1/2}, \frac{-1}{2}(1-s)^{1/2}, \frac{1}{\sqrt{2}} \rangle$

$$\|\vec{r}\,'(s)\| = \sqrt{\frac{1}{4}(1+s) + \frac{1}{4}(1-s) + \frac{1}{2}}$$
$$= \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{2}}$$
$$= \sqrt{1}$$
$$= 1 \quad \checkmark$$

(b) Since $\vec{r}(s)$ is unit speed, $\kappa(s) = \|\vec{r}\,''(s)\|$ and $\vec{N} = \frac{\vec{r}\,''(s)}{\|\vec{r}\,''(s)\|}$. So $\vec{r}\,''(s) = \langle \frac{1}{4}(1+s)^{-1/2}, \frac{1}{4}(1-s)^{-1/2}, 0 \rangle$

$$\vec{r}\,''(0) = \langle \tfrac{1}{4}, \tfrac{1}{4}, 0 \rangle$$

$$\Rightarrow \kappa(0) = \|\langle \tfrac{1}{4}, \tfrac{1}{4}, 0 \rangle\| = \frac{\sqrt{2}}{4} = \frac{1}{2\sqrt{2}}$$

$$\vec{N}(0) = 2\sqrt{2}\langle \tfrac{1}{4}, \tfrac{1}{4}, 0 \rangle = \langle \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0 \rangle$$

(c) The osculating plane, which is the plane that the curve comes closest to lying in at $\vec{r}(s)$, passes through $\vec{r}(0) = \langle \frac{1}{3}, \frac{1}{3}, 0 \rangle$ and has normal vector the binormal vector $\vec{B} = \vec{T} \times \vec{N} = \langle \frac{1}{2}, \frac{-1}{2}, \frac{1}{\sqrt{2}} \rangle \times \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \rangle$

$$= \left\langle \frac{-1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}} \right\rangle$$

So since $\frac{-1}{2}\left(\frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3}\right) + 0\left(\frac{1}{\sqrt{2}}\right) = 0$, it is the plane $\frac{-1}{2}x + \frac{1}{2}y + \frac{1}{\sqrt{2}}z = 0$, or $\boxed{-x + y + \sqrt{2}z = 0}$.

Problem 2.

Recall that one parametrization of the cycloid, the path traced by a point on the edge of a wheel of radius one as the wheel rolls forward, is $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t \rangle$.

(a) [5pts.] Find the arclength of $\mathbf{r}(t)$ along the interval $0 \leq t \leq 2\pi$, that is, as the wheel rolls through one full circle. [You may find it helpful to recall the following half-angle identity: $1 - \cos t = 2\sin^2(\frac{t}{2})$.]

(b) [5pts.] At what times t the curve is the point on the edge of the wheel whose motion is parametrized by this cycloid moving with speed 1?

(a) $\vec{r}\,'(t) = \langle 1 - \cos t, \sin t \rangle$

$$\|\vec{r}\,'(t)\| = \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}$$

$$= \sqrt{2 - 2\cos t}$$

$$= \sqrt{2(1-\cos t)}$$

$$= \sqrt{4\sin^2(\tfrac{t}{2})}$$

$$= 2\sin(\tfrac{t}{2}) \leftarrow \text{Note } \sin(\tfrac{t}{2}) \geq 0 \text{ on } 0 \leq t \leq 2\pi,$$

$$\text{Arclength} = \int_0^{2\pi} 2\sin(\tfrac{t}{2})\,dt = -4\cos(\tfrac{t}{2})\Big|_0^{2\pi}$$

$$= -4[\cos\pi - \cos 0]$$

$$= -4[-1-1]$$

$$= 8$$

(b) Speed is $\|\vec{r}\,'(t)\|$.

$$1 = 2\sin(\tfrac{t}{2})$$

$$\tfrac{1}{2} = \sin(\tfrac{t}{2})$$

$\rightarrow$ Assuming $0 \leq t \leq 2\pi$,

$$\frac{\pi}{6} = \frac{t}{2} \quad \text{or} \quad \frac{5\pi}{6} = \frac{t}{2}$$

$$\frac{\pi}{3} = t \quad \text{or} \quad \frac{5\pi}{3} = t$$

Problem 3.

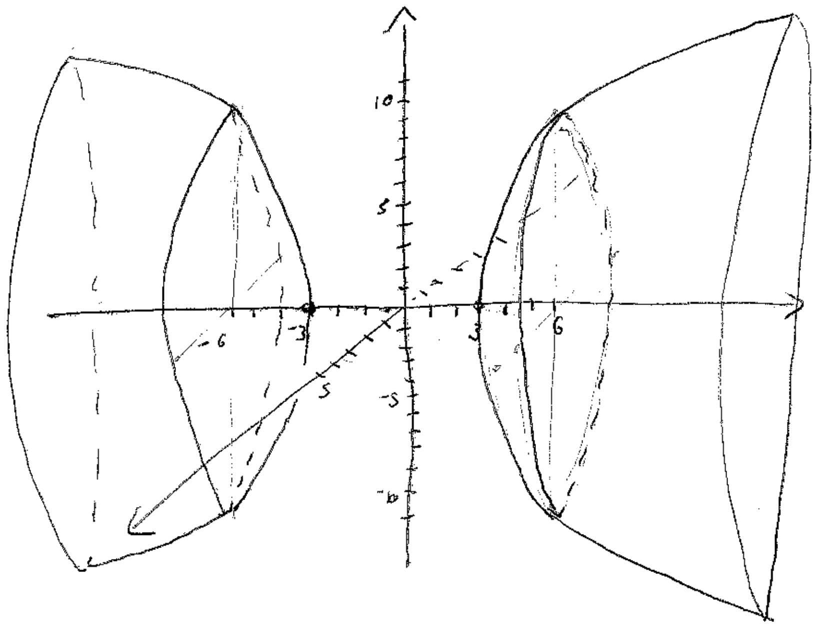
Draw the following.

(a) [5pts.] The quadric surface $9x^2 - 4y^2 + z^2 = -36$.

(b) [5pts.] The domain of the function $g(x, y, z) = \sqrt{16 - x^2 - 4y^2 - z^2}$.

(a) $9x^2 - 4y^2 + z^2 = -36$

$$\underbrace{-\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 + -\left(\frac{z}{6}\right)^2 = 1}_{\text{Hyperboloid of two sheets}}$$



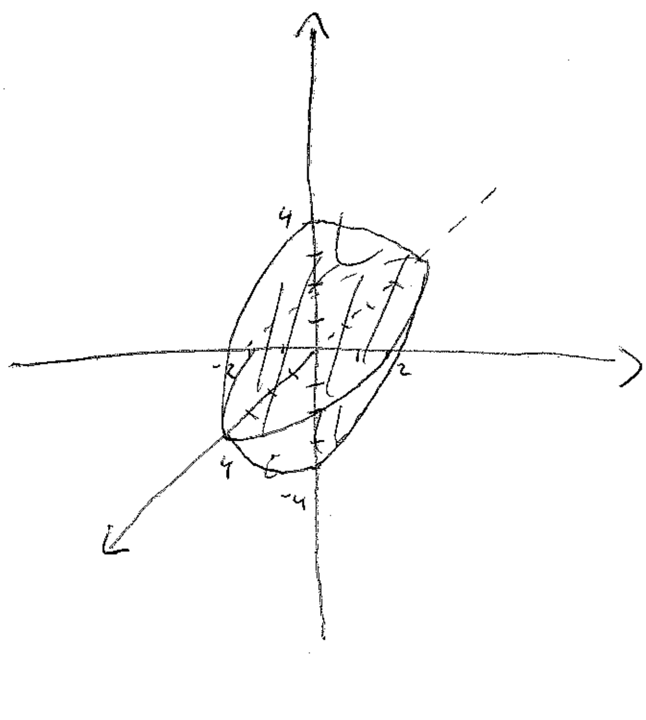
(b) Need $16 - x^2 - 4y^2 - z^2 \geq 0$

$16 \geq x^2 + 4y^2 + z^2$

$1 \geq \left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{4}\right)^2$

Everything inside the ellipsoid

$1 = \left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{4}\right)^2$ (including the ellipsoid itself.



Problem 4.

For each of the limits below, either compute the limit or prove that it does not exist. Justify your answers carefully.

(a) [3pts.] $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$.

(b) [3pts.] $\lim_{(x,y)\to(0,0)} y\tan^{-1}(\frac{1}{x^2})$.

(c) [4pts.] $\lim_{(x,y)\to(0,0)} xy^2\cos(e^{\frac{1}{x^2+50y^2}})$.

(a) We look at the limit in polar coordinates:

$$\lim_{r\to 0} \frac{r^2 \sin\theta\cos\theta}{\sqrt{r^2}} = \lim_{r\to 0} r\sin\theta\cos\theta$$

Now $0 \le |r\sin\theta\cos\theta| \le |r| = r$, and as a single variable limit $\lim_{r\to 0} r = 0$, so by the squeeze thm, $\lim_{r\to 0} \frac{r^2 \sin\theta\cos\theta}{r} = 0$ as well. Hence the limit is 0.

(b) Notice that $\lim_{(x,y)\to(0,0)} y = \lim_{y\to 0} y = 0$ and $\lim_{(x,y)\to(0,0)} \tan^{-1}(\frac{1}{x^2}) = \frac{\pi}{2}$.

So by the product law for limits, $\lim_{(x,y)\to(0,0)} y\cdot \tan^{-1}(\frac{1}{x^2}) = 0(\frac{\pi}{2}) = 0$.

(c). Notice, $0 \le |xy^2 \cos e^{\frac{1}{x^2+50y^2}}| \le |xy^2|$. Since $\lim_{(x,y)\to(0,0)} |xy^2| = 0$,

$\lim_{(x,y)\to(0,0)} |xy\cos(e^{\frac{1}{x^2+50y^2}})| = 0$, implying that $\lim_{(x,y)\to(0,0)} xy^2 e^{\frac{1}{x^2+50y^2}} = 0$.

Problem 5.

(a) [5pts.] Compute the first partial derivatives of $g(x,y,z) = xye^{z-x}$.

(b) [5pts.] Either give an example of a function $f(x,y)$ with partial derivatives $f_x(x,y) = 3y^2 - \sin x$ and $f_y(x,y) = 6y + \cos x$, or explain why one cannot exist.

(a) $g_x(x,y,z) = y\left[(1)e^{z-x} + x(-1)e^{z-x}\right]$

$= y(1-x)e^{z-x}$

$g_y(x,y,z) = xe^{z-x}$

$g_z(x,y,z) = xye^{z-x}$

(b) Note that $\frac{\partial}{\partial x}(f_y) = -\sin x$ and $\frac{\partial}{\partial y}(f_x) = 6y$. So $f_{xy} \neq f_{yx}$,

and by Clairaut's thm, such a function f cannot exist.